HIGHER FANO MANIFOLDS AND RATIONAL SURFACES

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Abstract. Let X be a Fano manifold of pseudo-index ≥ 3 such that $c_1(X)^2 - 2c_2(X)$ is nef. Irreducibility of some spaces of rational curves on X (in fact, a weaker hypothesis) implies a general point of X is contained in a rational surface.

1. Introduction

One consequence of the bend-and-break lemma is uniruledness of Fano manifolds, [MM86]. In fact, in characteristic 0, Fano manifolds are rationally connected, [KMM92], [Cam92]. We prove an analogous theorem with rational curves replaced by rational surfaces for Fano manifolds satisfying positivity of the second graded piece of the Chern character.

Definition 1.1. A Fano manifold is 2-*Fano* if $\mathrm{ch}_2(T_X)$ is nef, where $\mathrm{ch}_2(T_X)$ is the second graded piece of the Chern character, $\frac{1}{2}(c_1(T_X)^2 - 2c_2(T_X))$. In other words, $\deg(\mathrm{ch}_2(T_X)|_S)$ is nonnegative for every surface S in X.

Let $\mathcal{M}$ be a positive-dimensional, irreducible component of the Artin stack $\overline{\mathcal{M}}_{0,0}(X)$ of genus 0 stable maps to X whose general point of $\mathcal{M}$ parametrizes a stable map with irreducible domain. Denote by M the coarse moduli space of M. Denote by Δ the locally principal closed substack of $\overline{\mathcal{M}}_{0,0}(X)$ parametrizing stable maps with reducible domain. The closed substack $\mathcal{M} \cap \Delta$ is a Cartier divisor. The question we consider is uniruledness of M.

Theorem 1.2. *If X is 2-Fano, every point of M parametrizing a free curve and contained in a proper curve in $M - M \cap \Delta$ is contained in a rational curve in M.*

If a general point of M parametrizes a birational, free curve and is contained in a proper curve in $M - M \cap \Delta$, then a general point of X is contained in a rational surface.

The proof uses the bend-and-break approach of [MM86]. Given a general curve C in M, we need to bound the dimension of $\mathrm{Hom}(C, M)$ from below. Although M and $\mathcal{M}$ may be very singular, the deformation theory of stable maps nonetheless gives a useful lower bound. This uses Grothendieck-Riemann-Roch computations from [dJS05a]. Unfortunately, the formula has a negative term coming from intersection points of C and Δ. This is the reason for the hypothesis that $M - M \cap \Delta$ contains a proper curve. Luckily, there are nice sufficient conditions for $M - M \cap \Delta$ to contain many proper curves.

Proposition 1.3. *If the pseudo-index of X is ≥ 3 and every irreducible component of $\mathcal{M} \cap \Delta$ is an irreducible component of Δ, then $M - M \cap \Delta$ is a union of proper curves.*

This uses a contraction of the locally principal closed subspace Δ in $\overline{\mathcal{M}}_{0,0}(X)$ discovered in [CHS05] and independently by Adam Parker [Par05].

Section 3 gives some examples of 2-Fano manifolds and makes some observations about classification. Section 4 shows Theorem 1.2 is sharp in 2 ways. First, there are Fano manifolds that are not 2-Fano where the components $\mathcal{M}$ are not uniruled. Second, there are 2-Fano manifolds where the components $\mathcal{M}$ are uniruled but not rationally connected. Finally Section 5 speculates on sufficient conditions for the components $\mathcal{M}$ to be rationally connected.

2. Proof of the theorem

For every point x, denote by $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)_{\mathrm{nc}}$ the open subscheme of $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ parametrizing nonconstant morphisms.

Lemma 2.1. *The dimension of every irreducible component of Hom*$(\mathbb{P}^1, X, 0 \mapsto x)_{nc}$ *is at least as large as the pseudo-index of* X.

Proof. This follows from [Kol96, Theorem II.1.2, Corollary II.1.6]. □

Proof of Proposition 1.3. Let $f : X \hookrightarrow \mathbb{P}^N$ be a plurianticanonical embedding. Denote by $\overline{\mathcal{M}}_{0,0}(f) : \overline{\mathcal{M}}_{0,0}(X) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N)$ the associated embedding. Denote by $\phi : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N) \to Y$ the contraction of the boundary constructed in [CHS05]. Denote by N the image of $\mathcal{M}$ in Y.

Since the restriction of ϕ to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N) - \Delta$ is an open immersion, the restriction of $\phi \circ \overline{\mathcal{M}}_{0,0}(f)$ to $\mathcal{M} - \Delta$ is and immersion. Since $\mathcal{M}_{\mathrm{free}}$ is dense in $\mathcal{M}$, $\mathcal{M}$ has pure dimension equal to the expected dimension, and $\mathcal{M} \cap \Delta$ is a Cartier divisor. Therefore $\dim(N)$ equals $\dim(M)$ and $\dim(\mathcal{M} \cap \Delta)$ equals $\dim(M) - 1$.

If $i \leq j$, the restriction of ϕ to the boundary divisor $\Delta_{i,j}$ factors through the projection $\pi_j : \Delta_{i,j} \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, j)$. Denote $\Delta_{i,j} \cap \overline{\mathcal{M}}_{0,0}(X)$ by $\Delta_{X,i,j}$. Denote the restriction of π_j by $\pi_{X,j} : \Delta_{X,i,j} \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, j)$. By Lemma 2.1, every irreducible component of every fiber of $\pi_{X,j}$ has dimension ≥ 1, i.e., the difference of the pseudo-index and $\dim(\mathrm{Aut}(\mathbb{P}^1, 0))$. Therefore, for every irreducible component Δ' of Δ, the dimension of $\phi(\overline{\mathcal{M}}_{0,0}(f)(\Delta'))$ is strictly less than the dimension of Δ'. By hypothesis, every irreducible component Δ' of $\mathcal{M} \cap \Delta$ is an irreducible component of Δ. Since $\dim(\Delta')$ equals $\dim(\mathcal{M}) - 1$, the image of Δ' in N has dimension $\leq \dim(N) - 2$.

Since every connected component of Y is projective, also N is projective. Because $\dim(\mathrm{Image}(\Delta')) \leq \dim(N) - 2$, a general intersection of N with $\dim(N) - 1$ hyperplanes containing a point of $N - \mathrm{Image}(\Delta')$ is a complete curve that does not intersect $\mathrm{Image}(\Delta')$. Because there are only finitely many irreducible components of $\mathcal{M} \cap \Delta$, a general intersection of N with $\dim(N) - 1$ hyperplanes containing a point of $N - \mathrm{Image}(\mathcal{M} \cap \Delta)$ is a complete curve that does not intersect $\mathrm{Image}(\mathcal{M} \cap \Delta)$. The inverse image of this curve in $\mathcal{M} - \mathcal{M} \cap \Delta$ is a complete curve containing a given point of $\mathcal{M} - \mathcal{M} \cap \Delta$. □

Let C be a smooth, proper, connected curve and let $\zeta : C \to \overline{\mathcal{M}}_{0,0}(X) - \Delta$ be a nonconstant 1-morphism whose general point parametrizes a free curve of $(-K_X)$-degree e. Let B be a finite set of closed points of C. Denote by $(\pi : \Sigma \to C, F : \Sigma \to X)$ the associated family of stable maps.

Lemma 2.2. *The dimension at* $[\zeta]$ *of* $Hom(C, \overline{\mathcal{M}}_{0,0}(X), \zeta|_B)$ *is at least,*

$$deg(ch_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} deg(c_1(T_X)^2|_{F(\Sigma)}) + (e + dim(X) - 3)(1 - g(C) - \#(B)).$$

Proof. Consider the finite morphism $(\pi, g) : \Sigma \to C \times X$. Denote by $\mathcal{N}$ the cokernel of the map,

$$d(\pi, g) : T_\Sigma \to \pi^* T_C \oplus g^* T_X.$$

By a natural generalization of [Kol96, Theorem I.2.16], the dimension of $\mathrm{Hom}(C, \overline{\mathcal{M}}_{0,0}(X), \zeta|_B)$ at ζ is at least,

$$h^0(\Sigma, \mathcal{N}) - h^1(\Sigma, \mathcal{N}).$$

By the Leray spectral sequence, $h^2(\Sigma, \mathcal{N})$ equals $h^1(C, R^1\pi_*\mathcal{N})$. Because a general point of C parametrizes a free curve, the restriction of $\mathcal{N}$ to a general fiber of π is generated by global sections, thus has no higher cohomology. Thus $R^1\pi_*\mathcal{N}$ is a torsion sheaf so that $h^1(C, R^1\pi_*\mathcal{N})$ is 0. Therefore, the lower bound actually equals $\chi(\Sigma, \mathcal{N})$.

Finally, by the Grothendieck-Riemann-Roch computations from [dJS05a], $\chi(\Sigma, \mathcal{N})$ equals,

$$\deg(\mathrm{ch}_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e}\deg(c_1(T_X)^2|_{F(\Sigma)}) + (e + \dim(X) - 3)(1 - g(C) - \#(B)).$$

□

Proof of Theorem 1.2. Every proper curve in $M - M \cap \Delta$ is the image of a nonconstant 1-morphism $\zeta : C \to \mathcal{M} - \mathcal{M} \cap \Delta$ from a smooth curve C. The induced morphism $\mathrm{Hom}(C, \mathcal{M} - \mathcal{M} \cap \Delta) \to \mathrm{Hom}(C, M)$ is finite. By Lemma 2.2, $\dim(\mathrm{Hom}(C, M; \zeta|_B))$ behaves as if M is smooth along the image of ζ and the anticanonical degree of $\zeta(C)$ equals

$$\deg(\mathrm{ch}_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e}\deg(c_1(T_X)^2|_{F(\Sigma)}).$$

Because X is 2-Fano, this degree is positive. Therefore the usual bend-and-break argument applies, cf. [Kol96, Theorem II.5.8]. □

3. Examples of 2-Fano manifolds

All the results of this section, and more, are discussed and proved in the note [dJS05b]. There are a two families of 2-Fano manifolds. The first family comes from complete intersections. Let $\mathbb{P}$ be a weighted projective space of dimension n. Let $X \subset \mathbb{P}$ be a smooth complete intersection of type $(d_1, \dots, d_r)$. Then X is Fano if and only if $d_1 + \cdots + d_r \le n$. It is 2-Fano if and only if $d_1^2 + \cdots + d_r^2 \le n$.

The second family comes from Grassmannians. Let $\mathbb{G}$ be a Grassmannian $\mathrm{Grass}(k, n)$ of k-dimensional subspaces of a fixed n-dimensional vector space. Without loss of generality, assume $n \ge 2k$. This is Fano. It is 2-Fano if and only if either $k = 1$, $n = 2k$ or $n = 2k + 1$.

There are two operations for producing new 2-Fano manifolds. First, if X and Y are each 2-Fano, then the product $X \times Y$ is 2-Fano. The second operation is more interesting. Let X be a smooth Fano manifold and let L be a nef invertible sheaf. The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_X \oplus L^\vee)$ is Fano if and only if $c_1(T_X) - c_1(L)$ is ample. Assuming it is Fano, it is 2-Fano if and only if $\mathrm{ch}_2(T_X) + \frac{1}{2}c_1(L)^2$ is nef. Notice, it is not necessary that $\mathrm{ch}_2(T_X)$ is nef, i.e., X need not be 2-Fano.

There are other operations on Fano manifolds. It is reasonable to ask which of these produce 2-Fano manifolds. For instance, a projective bundle $\mathbb{P}(E)$ of fiber dimension ≥ 2 over a Fano manifold is also Fano if E satisfies a weak version of stability. However, if $\mathbb{P}(E)$ is 2-Fano then the pullback of E to every curve is a semistable bundle. If X is $\mathbb{P}^n$, for instance, this implies $\mathbb{P}(E)$ is simply $\mathbb{P}^m \times \mathbb{P}^n$. This, and other examples, suggest the following principle: an operation on Fano manifolds produces a 2-Fano manifold only if some vector bundle associated to the operation is semistable.

4. The theorem is sharp

The theorem is sharp in 2 ways. First, let X be a general cubic hypersurface in $\mathbb{P}^5$. This is Fano, but it is not 2-Fano. By the main theorem of [dJS04], there are infinitely many non-uniruled irreducible components $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(X)$ satisfying the hypotheses of Theorem 1.2.

Second, let Y be the $\mathbb{P}^1$-bundle over X, $Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_{\mathbb{P}^5}(-2)|_X)$. By the construction in the last section, Y is 2-Fano. Associated to the projection $\pi : Y \to X$, there is a 1-morphism $\overline{\mathcal{M}}_{0,0}(\pi) : \overline{\mathcal{M}}_{0,0}(Y) \to \overline{\mathcal{M}}_{0,0}(X)$. For an irreducible component $\mathcal{N}$ of $\overline{\mathcal{M}}_{0,0}(Y)$ containing a free curve, it is easy to prove the boundary $\mathcal{N} \cap \Delta$ is contracted. (However it is not true that every component of $\mathcal{N} \cap \Delta$ is a component of Δ.) Thus Theorem 1.2 implies N is uniruled. In fact, the restriction of $\overline{\mathcal{M}}_{0,0}(\pi)$ to $\mathcal{N}$ is birational to a projective bundle over the image component $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(X)$. Choosing $\mathcal{N}$ appropriately, $\mathcal{M}$ is one of the infinitely many non-uniruled irreducible components of $\overline{\mathcal{M}}_{0,0}(X)$. Therefore N is not rationally connected, and the MRC quotient of N is precisely M.

5. Speculation

For the counterexample Y in the previous section, $\mathrm{ch}_2(T_Y)$ is nef. But it is not "positive". It has intersection number 0 with the surface $\pi^{-1}(B)$ for every curve B in X. If X is a Fano manifold such that $\mathrm{ch}_2(T_X)$ has positive intersection number with every surface, is $\mathcal{M}$ rationally connected? We know no counterexample.

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